Elastic Interfaces on Disordered Substrates: From Mean-Field Depinning to Yielding

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(Received 30 May 2019; published 19 November 2019)

We consider a model of an elastic manifold driven on a disordered energy landscape, with generalized long range elasticity. Varying the form of the elastic kernel by progressively allowing for the existence of zero modes, the model interpolates smoothly between mean-field depinning and finite dimensional yielding. We find that the critical exponents of the model change smoothly in this process. Also, we show that in all cases the Herschel-Buckley exponent of the flow curve depends on the analytical form of the microscopic pinning potential. Within the present elastoplastic description, all this suggests that yielding in finite dimensions is a mean-field transition.

DOI: 10.1103/PhysRevLett.123.218002

Statistical physics is built on analogies. The comparison of typically complex problems with a small number of simpler ones for which an exact solution is known is the first step in many argumentative constructions. For instance, out-of-equilibrium phase transitions have been discussed in the mirror of equilibrium phenomena. The problem of depinning of an elastic manifold moving on a disordered landscape has been rationalized by analogy with the theory of critical phase transitions [1,2] and studied for more than thirty years already. Once this problem has been reasonably understood, it served in turn as the base model for a new analogy step. Depinning has shaped the theoretical endeavors in the understanding of the yielding transition of amorphous solids under deformation that received the full attention of the statistical physics community only recently. The problem with analogies is that, sometimes, they may prevent us from seeing the big picture.

Members of the family of sand-pile problems, both depinning and yielding are paradigmatic examples of driven transitions and are intuitively very alike. Depinning is related to the movement of an elastic manifold in the presence of a quenched disordered potential, under an external driving force. Yielding pertains to the flow of an amorphous solid upon the application of an external driving stress or deformation. In both cases, if the driving force is weak (and the possibility of thermal activation is excluded), the system remains in a frozen configuration; however, if a critical threshold is exceeded, the system reaches a dynamical state with a nonzero average velocity (depinning picture) or strain rate (yielding picture). The critical threshold defines the transition. In depinning the velocity-force characteristics of the system shows singular behavior at a critical force f_c. While $v = 0$ for $f < f_c$, the velocity behaves as $v \sim (f - f_c)^\beta$ when f increases above f_c, with β a well-defined number known as the flow exponent. In yielding the transition is characterized by the critical behavior of the strain rate $\dot{\gamma}$, which is zero when the stress σ is below a critical value σ_c, and becomes $\dot{\gamma} \sim (\sigma - \sigma_c)^\beta$ when $\sigma > \sigma_c$. The value of β is referred to again as the flow exponent. Its inverse $n \equiv 1/\beta$ is known as the Herschel-Buckley exponent.

The depinning transition finds a continuous model approach in the quenched Edwards-Wilkinson equation, which allows for analytical treatment using functional renormalization group (FRG) analysis. For yielding, elastoplastic models (EPMs) built at a coarse-grained scale [3] provide a similar description [4–6]. Nevertheless, despite the analogous construction, a FRG treatment of EPMs has found limitations; analytical support for a theory of yielding is only provided so far by mean-field variants [7–14]. In EPMs the instantaneous values of stress and plastic strain are evolved consistently. Under a condition of uniform load, the stress increases uniformly. When the stress locally exceeds a threshold, the local plastic strain increases at that patch, causing a reduction of the local stress and a perturbation of the stress in every other point in the system, following the action of elastic interactions. The form of these interactions is the one prescribed by the Eshelby propagator [15,16] of continuum mechanics which in d dimensions has a $1/r^d$ spatial decay and thus it is a long-range interaction. Also, it has alternating signs depending of the direction, with a quadrupolar symmetry. This anisotropy is a curse for the FRG approach and is responsible for special avalanche correlations in the form of slip lines, that greatly determine the differences between yielding and depinning.

In this Letter we show that mean-field depinning and yielding transition in finite dimensions can be considered to be special cases of a generalized mean-field problem and, therefore, described within the same framework. Simply considering an elastic kernel as the sum of two contributions $G^{\rm MFD}$ and $G^{\rm Y}$, corresponding, respectively, to a constant value propagator (in Fourier space) and the Eshelby propagator, we are able to *smoothly* interpolate $[\varepsilon G^{\rm MFD} + (1-\varepsilon)G^{\rm Y}$ with $0 \geq \varepsilon \geq 1]$ between mean-field depinning and yielding. In particular, we observe a smooth transition in the values of the critical exponents between the two limiting cases. Thus, our work suggests an alternative view for the theoretical tackling of the yielding transition in any dimension, interpreting it as a particular case of a general mean-field problem that includes itself the fully connected mean-field (MF) depinning.

A general model for MF depinning and yielding.—The model that allows us to describe MF depinning and yielding on the same footing is constructed in the following way. A scalar field e_i is defined on the sites i of a d dimensional ordered lattice. For depinning e_i represents the interface position at site i, whereas for yielding e_i is the strain of an elemental volume of the system at site i. The dynamics is described by overdamped equations of motion:

$$\eta \frac{de_i}{dt} = f_i(e_i) + \sum_j G_{ij} e_j + \sigma. \tag{1}$$

In the case of depinning, the terms $f_i(e_i) \equiv -dV_i/de_i$ represent the force exerted by the external pinning potential V_i on the interface, whereas for yielding they describe the local stress thresholding behavior of a small piece of the amorphous material. In both cases, the form of V_i is similar: they have minima at different e_i positions representing local equilibrium states. G_{ij} represents the elastic interaction between e values at different points. We restrict to cases in which this interaction preserves the homogeneity of the system, then G_{ij} depends only on the difference between the (vector) positions i and j. Also, $G_{ij} = G_{ji}$ is assumed. Elastic forces should be balanced in the system; therefore $\sum_i G_{ij} = 0$ must be also satisfied. This still leaves us a lot of freedom in the choice of a general form for G_{ij}. Nevertheless, an important additional constraint must be fulfilled: in the absence of local forces ($f_i \equiv 0$) the flat configuration of the interface $e_i = \text{cst}$ must be stable. This condition becomes more transparent in Fourier space, where Eq. (1) reads (for $\mathbf{q} \neq 0$)

$$\eta \frac{de_{\mathbf{q}}}{dt} = f(e)|_{\mathbf{q}} + G_{\mathbf{q}} e_{\mathbf{q}}. \tag{2}$$

The stability condition is then $G_{\mathbf{q}} \leq 0$.

In the following we will mainly discuss the interaction kernel in Fourier space. One can consider "generalized mean-field models" defined as cases in which the $G_{\mathbf{q}}$ is zeroth order homogeneous in $|q|$. These kernels produce a function G_{ij} that is either independent of distance or decaying with r_{ij} as r_{ij}^{-d}. In both cases, the effect of a single site onto another site is negligible compared to the combined effect of all other sites in the lattice. Therefore, the dynamics of a given site can be solved by considering the existence of a (fluctuating) prescribed external field (see Ref. [13]). In particular, the forms of $G_{\mathbf{q}}$ for mean-field depinning and yielding satisfy the prescription just mentioned. For mean-field depinning $G_{\mathbf{q}}^{\rm MFD} = -1$ for $\mathbf{q} \neq 0$, whereas for yielding $G_{\mathbf{q}}$ is the Eshelby propagator that in two dimensions can be written as ($\mathbf{q} \neq 0$)

$$G_{\mathbf{q}}^{Y} = -\frac{(q_x^2 - q_y^2)^2}{(q_x^2 + q_y^2)^2}. \tag{3}$$

In both cases $G_{\mathbf{q}=0}$ is taken as zero in a stress conserved dynamics, as it follows from the condition $\sum_i G_{ij} = 0$. The evolution of the uniform mode in Eq. (1) is thus directly obtained by spatial averaging as (we set $\eta \equiv 1$ in the following)

$$\dot{\gamma} \equiv \frac{d\overline{e_i}}{dt} = \overline{f_i(e_i)} + \sigma, \tag{4}$$

which defines the global strain rate $\dot{\gamma}$. The fact that both $G^{\rm MFD}$ and $G^{\rm Y}$ share the property of being $\mathcal{O}(q^0)$ allows us to believe that mean-field depinning and yielding may experience many common features.

Concerning the properties of the disorder term $f_i(e_i)$, we restrict to the case of locally correlated potentials, where the ensemble average $\langle V_i(z)V_i(z+\Delta)\rangle$ decays to zero sufficiently fast with Δ. Also, we consider the disorder site by site to be totally uncorrelated, namely, $\langle V_i(z)V_j(z)\rangle = 0$ for $i \neq j$. With these correlation properties, renormalization group theory teaches that the detailed form of $V_i(z)$ should be irrelevant when determining the critical properties of the transition, as long as the elastic interaction G_{ij} decays sufficiently fast in space as a function of $r \equiv |r_i - r_j|$, concretely, if $G_{ij} \sim r^{-\alpha}$ with $\alpha > d$. In cases in which $\alpha \leq d$ (there included mean-field depinning and yielding) this result does not apply, and different values are obtained for the dynamical exponents when considering "cuspy" or "smooth" potentials [1,13,17,18]. We will mainly focus on the case of a cuspy form for the pinning potential, taking V_i as composed by a concatenation of parabolic pieces. A brief consideration of the smooth potential case is included by the end.

Results.—We present simulations in two dimensions using a kernel

$$G_{\mathbf{q}} \equiv (1-\varepsilon)G_{\mathbf{q}}^{\rm Y} + \varepsilon G_{\mathbf{q}}^{\rm MFD} \tag{5}$$

that interpolates between the mean-field depinning case (for $\varepsilon = 1$) and the yielding case (for $\varepsilon = 0$). Notice that this interpolation corresponds to a variation of the angular dependence of the kernel (that for any ε is independent of $|\mathbf{q}|$) that is essentially different from tuning the algebraic

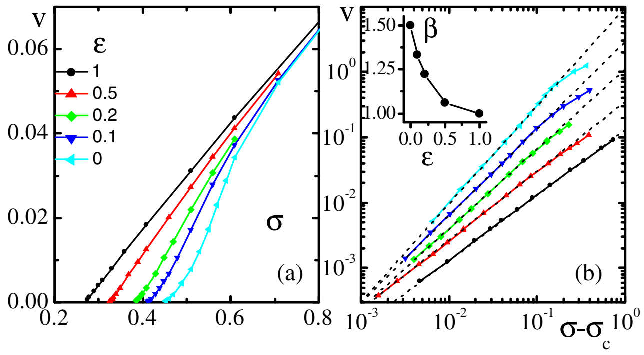

FIG. 1. Flow curves for different ε interpolating between mean-field depinning ($\varepsilon = 1$) and yielding ($\varepsilon = 0$) in the linear (a) and logarithmic scale (b). The inset shows the values of β determined as the slope of the straight lines, for $\sigma \to \sigma_c$. In (b) curves are displaced vertically to facilitate visualization. System size is $N = 512^2$.

decay of the elastic interactions as in Ref. [19]. The same linear combination of kernels was used in Ref. [20] to analyze the depletion of soft modes suffered by a "modified" Eshelby, but the critical properties of the combined model were not addressed. The stress-controlled and quasistatic strain-controlled protocols to determine the flow curves and the avalanche statistics are described in the Supplemental Material [21].

Both for an elastic interface undergoing depinning and for an amorphous solid at the onset of yielding, a singular behavior of $\dot{\gamma}$ (strain-rate or velocity) is expected at σ_c. Figure 1 shows the flow curves for different values of ε. By plotting the data in logarithmic scale close to $(\sigma - \sigma_c)$ [22], a clear power-law behavior allows us to determine the flow exponent β. Going from $\varepsilon = 1$ (MFD) to $\varepsilon = 0$ (Y) we observe that the exponent moves from $\beta = 1$ to $\beta \simeq 1.5$. Notably, this variation is *smooth*, as the inset in Fig. 1(b) shows, indicating the continuous evolution that exists between mean-field depinning and yielding.

We now discuss the avalanche size distribution $P(S)$ associated to the transition. When $P(S)$ is taken from large collections of avalanches obtained in a quasistatic simulation, it is expected to be power-law distributed, namely, $P(S) \sim S^{-\tau} f(S/S_{\max})$ with the cutoff function $f(x)$ behaving as $f_{x\to 0} \to 1$ and $f_{x\to\infty} \to 0$, and $S_{\max}$ depending on the system size L and the stress nonconserving parameter κ used to define the value of $G_{\mathbf{q}=0}$ (see Supplemental Material [21]). Avalanche-size distributions are shown in Fig. 2 for different ε [23]. In depinning mean-field models $\tau = 3/2$, and, in fact, we obtain $\tau \simeq 1.5$ when $\varepsilon = 1$. But as we decrease ε moving towards yielding, τ diminishes, becoming $\tau \simeq 1.33$ at $\varepsilon = 0$, in agreement with previous numerical simulations [3,5,18,24,25] [26]. Surprisingly, this change is *continuous*: the avalanche size distribution critical exponent τ is a smooth function of the parameter ε.

Directly related to the avalanche mean size is the loading stress needed to trigger avalanches $x_{\min}$. It is known

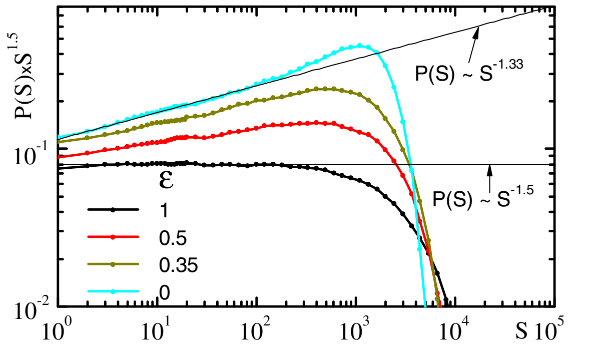

FIG. 2. Avalanche distributions $P(S)$ interpolating between mean-field depinning ($\varepsilon = 1$) and yielding ($\varepsilon = 0$). Note the particular scaling of the vertical axis to emphasize differences (curves vertically displaced for clarity). System size is $N = 512^2$.

for yielding [5,27,28] that its mean value scales subextensively with system volume $N = L^d$: $\langle x_{\min}\rangle \propto N^{-\alpha}$, with $0 < \alpha < 1$. This phenomenological subextensiveness in the plastic flow of amorphous solids under deformation was interpreted [5,29] as a consequence of a peculiar shape for the steady state distribution $P(x)$ of local distances to threshold x [30]. If this quantity has the form $P(x) \sim x^{\theta}$ as $x \to 0$, one can deduce [28] that $\langle x_{\min}\rangle \propto N^{-1/(1+\theta)}$. Then, $\theta = 0$ is expected for depinninglike models (where the kernel G_{ij} is non-negative) and $\theta > 0$ for yielding models (where the kernel G_{ij} alternates in sign). Figure 3 shows results for $\langle x_{\min}\rangle$ vs L for systems with different ε. Power-law fits allow for a precise determination of the exponent values. Consistently with the expectation, $\theta = 0$ for MF depinning ($\varepsilon = 1$) and a strictly positive value for the yielding case ($\varepsilon = 0$). What is surprising again is that θ turns out to be a *continuous* function of the crossover parameter ε, going from 0 to $\simeq 0.5$ as we move from the MF-depinning limit to the 2D-yielding case, as displayed in Fig. 3 inset. This tells us that we are dealing with a family of similar problems, each of them characterized by a given degree of subextensiveness of the load needed to trigger new avalanches.

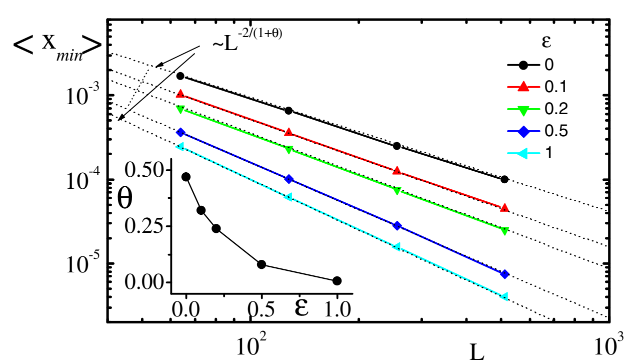

FIG. 3. Evolution of $\langle x_{\min}\rangle$ as a function of linear system size L, for different values of ε. Values of θ obtained by fitting straight lines of slope $-d/(1+\theta)$ are plotted in the inset.

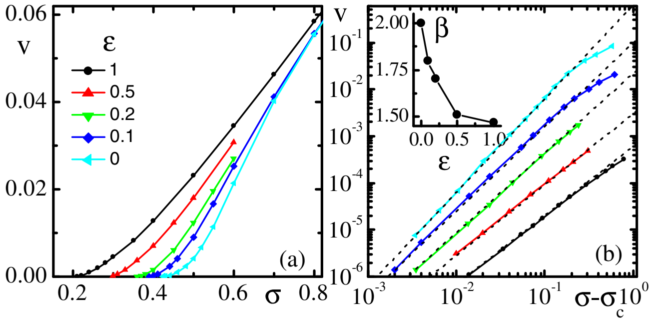

FIG. 4. Same as Fig. 1 but using a smooth pinning potential. The values obtained for β at corresponding values of ε are found to be $1/2$ larger than those for cuspy potentials.

Smooth pinning potentials.—All results presented so far were obtained using a local disorder potential that has cusps in the transition from one potential well to the next one. Usually, according to renormalization arguments, this kind of detail on the microscopic potential should not influence the critical properties of a system. In particular, the critical exponents of the depinning transition are expected to be independent of the potential being of the cuspy or smooth type. Nevertheless, the fully connected mean-field case is an exception (see discussion in Ref. [17]). There, we know that depinning displays a value $\beta = 1$ for cuspy pinning potentials and a value $\beta = 3/2$ for smooth pinning potentials.

The smooth crossover of exponents that we observe between mean-field depinning and yielding in Fig. 1 suggests that we will also find the above described dichotomy in the flow curve exponent value for the yielding case. Even more, we can expect to find larger values of β using smooth potentials for any value of the crossover parameter ε. Results of simulations contained in Fig. 4 confirm this expectation. Note that the β value for smooth potentials always (i.e., for each ε) exceeds in $1/2$ the one for cuspy potentials, in full agreement with recent theoretical expectations derived from the Prandtl-Tomlinson model under stochastic driving [14].

Three and larger dimensional cases.—In $d = 3$, the Eshelby kernel for one scalar component of the deviatoric strain in Fourier space can be written as

$$G_{\mathbf{q}}^{3\mathrm{D}} = \frac{2q_x^2(q_y^2 + q_z^2)}{(q_x^2 + q_y^2 + q_z^2)^2} - 1. \qquad (6)$$

One can notice that again $G_{\mathbf{q}} \sim \mathbf{q}^0$, and this is true in general for $d > 1$. Therefore, we expect all our numerical observations and conclusions obtained in $d = 2$ to be valid also in $d = 3$ and higher dimensions. For instance, preliminary simulations using the previous form of the Eshelby kernel in $d = 3$ with the cuspy potential display again a smooth crossover between $\beta^{\mathrm{Y}} \simeq 1.3$ (not far from other estimations [5]) and $\beta^{\mathrm{MFD}} = 1$ continuously moving with ε. The value of the pseudogap exponent θ also changes continuously with ε. The reduction of β in passing from two to three dimensions can be rationalized as a consequence of the reduced density of zero modes in the elastic propagator in $d = 3$ as compared to the $d = 2$ case.

Why a smooth exponent crossover is surprising?— The fact that critical exponents (β, θ, and τ in particular) vary smoothly when interpolating between MFD and Y is remarkable. This situation is not expected in general when studying crossovers between different asymptotic behaviors. Consider, for instance, the case of long-range depinning. Choosing a kernel decaying in space as $G_1 \sim 1/r^{\alpha_1}$ ($d < \alpha_1 < d + 2$) a set of critical exponents is obtained. For other decaying forms of the kernel $G_2 \sim 1/r^{\alpha_2}$ the exponents are different. However, if we combine the two kernels in the form $G = (1 - \varepsilon)G_1 + \varepsilon G_2$ the system will display the critical behavior corresponding to the lowest value of α. In other words, if two *different* criticalities are mixed together the system will display at long enough scales the critical exponents corresponding to the longest range interactions. In order to have a variation of the critical exponents with ε, the long range weight of G_1 and G_2 must be similar. This noncommon case is what happens in our kernel combination, clearly manifested in the $\mathbf{q}$ space form of the propagator: both the mean-field depinning constant kernel and the Eshelby kernel of yielding scale as $\mathbf{q}^0$. The interactions in real space at distances of the order of the system size are $\mathcal{O}(1/L^d)$ in both cases.

Conclusions.—We have studied a mesoscopic implementation of a scalar model for a generalized elastic manifold on a disordered landscape that is able to describe both depinning in mean-field and finite dimensional yielding by changing the form of the elastic kernel interaction. The most important result is the observation of a *smooth* transition between MFD and Y, as the kernel interpolates linearly between the two limiting cases. The identification of a common scenario for both transitions is assisted by recent reports of phenomenological properties akin to mean-field depinning in numerical simulations of yielding [13,18,31]. In particular, dynamical critical exponents are seen to depend on the details of the local disorder potential, a well-known fact for mean-field depinning [1,17].

In recent years, different attempts have been made to address the yielding transition of amorphous solids from an analytical perspective. Nevertheless, the noncompliance with the hypothesis needed for the FRG analysis has largely confined these treatments to mean-field Hébraud-Lequeux-like approaches [7–10] and heavy-tail noise variants [11–14]. Those studies provided a common general picture but did not forge a consensus about critical exponents and scaling laws. One of the main conclusions of the present work is that the yielding transition, as described by a scalar model with an Eshelby interaction, can be treated as a special case of a generalized mean-field problem which has the very well-known fully connected MF-depinning problem as one limiting case.

Our work suggests therefore that, instead of focusing on nontrivial correlations depending on the propagator properties and the dimension, a strategic angle of attack for theoretical studies of the yielding transition could be to start from a fully connected depinning system and explore perturbations of the Eshelby type to the constant propagator.

E. E. F. acknowledges support from Grant No. PICT 2017-1202, ANPCyT (Argentina).

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[22] σ_c depends on the value of ε and must be fitted for each curve.
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